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Abstract

Standard economic models of groundwater management assume perfect transmissivity (i.e., the aquifer behaves as a bathtub), no external effects of groundwater stocks, and/or homogenous agents. In this article, we develop a model relaxing these assumptions. Although our model generalizes to an arbitrary number of cells, we are able to obtain key insights with a two-cell finite-horizon differential game. We find a simple linear mechanism that induces the socially optimal extraction path in Markov-perfect equilibrium. Moreover, implementation requires that the regulator need only monitor the state of the resource (e.g., depth of the aquifer), not individual extraction rates. We illustrate the mechanism with a simulation based on data from the Indian state of Andhra Pradesh. The simulation suggests that significant welfare loss may occur if the regulator disregards physical and economic complexity.

Keywords: groundwater, differential games, imperfect monitoring

Subject Area Classification: Water Resources

1 Introduction

Over the past four decades, a large literature has developed on optimal aquifer management. An important assumption encompassing used in much of this work is that an aquifer behaves like a one-dimensional "bathtub." In a bathtub model, water flows to the lowest point instantaneously and the water table is level throughout. Despite this assumption's mathematical convenience, aquifers are not underground caves filled with water, but rather saturated materials such as porous rock. As a result, transmissivity (horizontal flow) is lower than in a bathtub and recharge rates may vary across space.[1]

In a bathtub model, spatial considerations are unimportant. With limited transmissivity, however, location matters. Cones of depression develop around individual wells, and the impact of extraction on other users decreases in distance from individual wells. Water-extracting agents are not uniformly located on the land overlying the water reserves; instead, they tend to be found in discrete clusters. Thus, a modeling assumption ignoring these effects can be expected to yield results of questionable validity.

Literature abandoning the bathtub model in favor of more realistic dynamics has avoided strategic interaction among agents [5, 7, 31], and/or has assumed identical agents [15, 11].[2] In addition, while these studies compare unregulated equilibrium outcomes with socially optimal outcomes, they do not derive policy mechanisms to attain the social optimum.

The importance of accounting for spatial complexity and heterogeneous agents becomes even greater when equity considerations are brought into play. In developing countries, agriculture is often characterized by large land holdings of relatively wealthy owners and small tracts worked by poorer households. Analysis of policy welfare implications in such a setting requires a model of water extraction that approximates actual water table dynamics while allowing for strategic behavior among heterogeneous agents.

Our model also allows for another element overlooked in the previous literature: stock

[1] See Brozovic et al. [5] for a thorough discussion of these issues.

[2] Assuming homogeneous agents greatly simplifies the analysis. All agents behave symmetrically, extracting the same amount of water in each period. There are thus no horizontal flows and hydrology is irrelevant.

externalities. Groundwater extraction does not necessarily take place in environmental or political-economic isolation. The level of the groundwater stock may have costs felt beyond the users themselves. Environmental impacts may include effects on nearby wetlands, land subsidance, or saltwater intrusion in coastal areas. Political effects (our present focus) may be felt if farmers' activities are subsidized by the state. For example, it is common in many developing countries (India is a prominent case) for agricultural electricity to be provided either for free or with a lump-sum tariff (see Dubash [10], Shah [26], and a recent World Bank report [29]). In general, the lower the water table, the more energy is required to extract the water. A stock externality is thus generated to the extent that electricity costs are borne by the state and need to be financed by distortionary taxes.

In this paper we adopt a spatially and economically complex model of groundwater extraction addressing the above considerations. Within this environment, we derive competitive equilibrium and socially-optimal extraction paths and investigate a simple class of policy instruments. Specifically, since continuous monitoring of agent extraction rates is likely to be infeasible, we only allow governmental transfers to be made on the stock of the resource. In this sense, the regulatory problem is similar to that of a non-point source pollution setting (e.g., [24, 30]). As exceedingly complex instruments are unlikely to be appealing in practice, we restrict attention to simple linear additive mechanisms (e.g., a linear tax based on water table depth). Interestingly, imposing such severe restrictions on policy instruments has no adverse effect on social welfare. We exhibit a simple mechanism that exactly induces the socially optimal extraction path in Markov-perfect equilibrium.

To illustrate the importance of these modeling techniques we conduct comparative simulations based on aquifer characteristics in the Indian state of Andhra Pradesh. In a simple two player game we derive the socially optimal extraction path and exhibit the linear transfer schemes that induce it in equilibrium. We further investigate the policy implications of improperly using a bathtub model and of failing to allow for agent heterogeneity. Our findings suggest that significant welfare loss may result from implementing policy that is predicated on incorrect physical and technological assumptions.

The paper is organized as follows. Focusing initially on the two-cell case, in Sections 2 and 3 we introduce our model and analyze the socially optimal solution. In Section 4 we discuss simple policy tools and prove that they can induce any feasible and continuously differentiable extraction path in Markov-perfect equilibrium, including the social optimum. In Section 5 we conduct a numerical simulation that applies our theoretical insights to a simple groundwater game set in rural India. We conclude in Section 6. The Appendix extends the analysis to an arbitrary number of cells and internalized energy costs.

2 The Model

Consider two adjacent aquifer cells, indexed by $i = 1, 2$, having n_i agents. For expository reasons, we assume $n_1 = n_2 = 1$, so agents and cells are interchangeable.[3]

Let $x_i(t)$, $q_i(t)$, and $h_i(q_i(t))$ denote water table elevation, extraction rate, and profit for agent i at time t. Profit depends only on extraction: All stock effects are external to the agents.[4] The profit function is twice continuously differentiable and strictly concave.

Departing from the one-dimensional bathtub model, the water tables of agents 1 and 2 follow the set of differential equations (a dot indicates a derivative with respect to time):

$$a_i \cdot \dot{x}_i = r - q_i + c[x_j - x_i], \quad i = 1, 2 \;\; i \neq j.^5 \tag{1}$$

Variations of these dynamics appear in [11, 15, 31]. Here, r is the (uniform) rate of recharge, and a_i is the surface area of agent i's land multiplied by the storativity coefficient of agent i's micro-watershed (which depends on unspecified geological factors). Parameter c is the transmissivity between adjacent aquifer cells and is a measure of the "connectivity" between the

[3] If $n_i = 1$ for all i, it is theoretically possible to infer the pumping schedules of agents from the evolution of the water tables. In the more general case, however, this is no longer true.

[4] Due, for example, to subsidized electricity costs. We note, however, that our analysis extends to the case where agents incur energy costs in the sense of Gisser and Sanchez [12] and Rubio and Casino [22]. That is, where agents pay an energy cost that is proportional to the product of their pumping rates times water table drawdown. We discuss this case in the Appendix.

agents. The term $c[x_j - x_i]$ is then the water flux between agent i and j's micro-watersheds.[6]

Solving the system of differential equations given by (1) yields water table levels at time t, as functions of initial conditions, x_i^0, and the extraction history $q_i(s)$ for $0 \leq s \leq t$. Letting $c_i \equiv c/a_i$, agent i's water table level at time t is:

$$\begin{aligned} x_i(t) \quad = \quad & \frac{1}{a_1 + a_2}\Big\{ x_i^0 \left[a_i + a_j \mathrm{e}^{-[c_1+c_2]t}\right] + x_j^0 a_j \left[1 - \mathrm{e}^{-[c_1+c_2]t}\right] + \\ & \int_0^t \left[[r - q_i(s)] \left[1 + \frac{a_j}{a_i}\mathrm{e}^{[c_1+c_2][s-t]}\right] + [r - q_j(s)]\left[1 - \mathrm{e}^{[c_1+c_2][s-t]}\right]\right] \mathrm{d}s \Big\}. \end{aligned} \tag{2}$$

Lemma 1 shows how these dynamics nest the extreme cases of unconnected cells and a single-cell if the extraction rate is bounded and cannot change too quickly.

Lemma 1 *Let $q_i(t;c)$ and $x_i(t;c)$ denote the extraction rate and water table at time t as functions of c. If extraction paths $q_i(t)$ are bounded and differentiable with bounded derivatives, water table dynamics defined by Eq. (1): (i) approach $\dot{x}_i = \frac{r - q_i(t;0)}{a_i}$ as $c \to 0$; and (ii) approach $\dot{x}_i = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1 + a_2}$ as $c \to \infty$.*

Proof. (i) Letting $\lim_{c \to 0} q_i(t;0) = q_i(t;0)$, the dynamics of an unconnected aquifer are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lim_{c\to 0} x_i(t;c)\right] = \lim_{c \to 0}\frac{\mathrm{d}}{\mathrm{d}t}\left[x_i(t;c)\right] = \frac{r - q_i(t;0)}{a_i}, \text{ for all } t \in [0,T]. \tag{3}$$

Taking limits as $c \to 0$, Eqs. (1) and (2) arrive at this expression.

(ii) Letting $\lim_{c\to\infty} q_i(t;c) = q_i(t;\infty)$, the dynamics of a single-cell aquifer are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\lim_{c\to \infty} x_i(t;c)\right] = \lim_{c \to \infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[x_i(t;c)\right] = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1 + a_2}, \text{ for all } t \in [0,T]. \tag{4}$$

Consider the water table equations $x_i(t,c)$ given by Eq. (2). Since the function $q_i(t,c)$ is bounded, the Bounded Convergence Theorem (see Rudin [23]) implies that the integral of

[6]Using water balance and connectivity between individual cells with uniform recharge rates to model flows is a simple version of the finite difference discretization methods commonly used in the hydrological literature to simulate existing aquifers with complex geometry and boundary conditions [14]. The $n \geq 2$ player version of our approach, detailed in the appendix, readily extends to these more general models.

the limit is equal to the limit of the integral. Taking the limit as $c \to \infty$ in Eq. (2) yields:

$$\lim_{c\to\infty} x_i(t;c) = \frac{x_i^0 + x_j^0 + 2rt - \lim_{c\to\infty}\int_0^t q_1(s;\infty) + q_2(s;\infty)\mathrm{d}s}{a_1+a_2}.$$

Differentiating with respect to t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lim_{c\to\infty} x_i(t;c)\right) = \frac{2r - q_1(t;\infty) - q_2(t;\infty)}{a_1+a_2}.$$

Substituting $x_i(t;c)$ and $x_j(t;c)$ from Eq. (2) into agent i's dynamics yields:

$$\begin{aligned}
a_i\dot{x}_i(t;c) &= r - q_i(t;c) + \tfrac{c}{a_1+a_2}\Big[\mathrm{e}^{-[c_1+c_2]t}\Big(x_j^0 a_j \Big(1+\tfrac{a_i}{a_j}\Big) - x_i^0 a_i \Big(1+\tfrac{a_j}{a_i}\Big)\Big) \\
&\quad + \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\Big(r\Big(\tfrac{a_i}{a_j} - \tfrac{a_j}{a_i}\Big) + q_i(s;c)\Big(1+\tfrac{a_j}{a_i}\Big) - q_j(s;c)\Big(1+\tfrac{a_i}{a_j}\Big)\Big)\mathrm{d}s\Big] \\
&= r\Big(\tfrac{2a_i}{a_1+a_2} - \tfrac{a_i^2-a_j^2}{[a_1+a_2]^2}\mathrm{e}^{-[c_1+c_2]t}\Big) \\
&\quad -q_i(t;c) + \tfrac{c}{a_1+a_2}\Big[\mathrm{e}^{-[c_1+c_2]t}\Big(x_j^0 a_j \Big(1+\tfrac{a_i}{a_j}\Big) - x_i^0 a_i \Big(1+\tfrac{a_j}{a_i}\Big)\Big) \\
&\quad + \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\Big(q_i(s;c)\Big(1+\tfrac{a_j}{a_i}\Big) - q_j(s;c)\Big(1+\tfrac{a_i}{a_j}\Big)\Big)\mathrm{d}s\Big] \\
&= r\Big(\tfrac{2a_i}{a_1+a_2} - \tfrac{a_i^2-a_j^2}{[a_1+a_2]^2}\mathrm{e}^{-[c_1+c_2]t}\Big) \\
&\quad -q_i(t;c) + \tfrac{c}{a_1+a_2}\mathrm{e}^{-[c_1+c_2]t}\Big(x_j^0 a_j \Big(1+\tfrac{a_i}{a_j}\Big) - x_i^0 a_i \Big(1+\tfrac{a_j}{a_i}\Big)\Big) \\
&\quad +\tfrac{a_j}{a_1+a_2}\Big(q_i(t;c) - \mathrm{e}^{-[c_1+c_2]t}q_i(0;c) - \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\tfrac{\mathrm{d}}{\mathrm{d}s}[q_i(s;c)]\mathrm{d}s\Big) \\
&\quad -\tfrac{a_i}{a_1+a_2}\Big(q_j(t;c) - \mathrm{e}^{-[c_1+c_2]t}q_i(0;c) - \int_0^t \mathrm{e}^{[c_1+c_2][s-t]}\tfrac{\mathrm{d}}{\mathrm{d}s}[q_j(s;c)]\mathrm{d}s\Big).
\end{aligned}$$

Taking the limit of this expression as $c \to \infty$ and applying the Bounded Convergence Theorem yields the desired result. ■

3 Social Optimum

A social planner wishes to maximize the net benefit of water extraction: the discounted (at rate $\delta > 0$) sum of agent profit less social damages (e.g., cost of energy used in extraction), denoted $D(\mathbf{q}(t), \mathbf{x}(t))$, where $\mathbf{q}(t) \equiv (q_1(t), q_2(t))'$ and $\mathbf{x}(t) \equiv (x_1(t), x_2(t))'$. This damage function is increasing in q_i, decreasing in x_i, and strictly convex in all its arguments. The terminal time is T, and the "scrap value" of the aquifer is $-D^T(\mathbf{x}(T))$. Initial conditions are $\mathbf{x}^0 \equiv (x_1^0, x_2^0)'$. The social planner's optimal control problem is

$$\begin{aligned} \max_{\mathbf{q}(t)} \quad & \int_0^T \mathrm{e}^{-\delta t} \left[\sum_{i=1}^{2} h_i(q_i(t)) - D(\mathbf{q}(t), \mathbf{x}(t)) \right] \mathrm{d}t - \mathrm{e}^{-\delta T} D^T(\mathbf{x}(T)) \\ \text{subject to:} \quad & a_i \dot{x}_i = r - q_i + c[x_j - x_i], \quad i, j = 1, 2; i = j \\ & \mathbf{q}(t) \geq \mathbf{0}, \quad \mathbf{x}(0) = \mathbf{x}^0. \end{aligned} \tag{5}$$

Letting $\boldsymbol{\lambda}(t) \equiv (\lambda_1(t), \lambda_2(t))'$ denote costate variables, the current-value Hamiltonian is:

$$\begin{aligned} H(\mathbf{q}(t), \mathbf{x}(t), \boldsymbol{\lambda}(t), t) \;\; = \;\; & \sum_{i=1}^{2} h_i(q_i(t)) - D(\mathbf{q}(t), \mathbf{x}(t)) \\ & + \sum_{i=1, j \neq i}^{2} \frac{\lambda_i(t)}{a_i} \left[r - q_i(t) + c[x_j(t) - x_i(t)] \right]. \end{aligned}$$

Necessary conditions for a maximum are

$$\begin{aligned} \frac{\mathrm{d}h_i(q_i(t))}{\mathrm{d}q_i} - \frac{\partial D}{\partial q_i} - \frac{\lambda_i(t)}{a_i} \quad & \leq \quad 0 \\ q_i(t) \quad \frac{\mathrm{d}h_i(q_i(t))}{\mathrm{d}q} - \frac{\partial D}{\partial q_i} - \frac{\lambda_i(t)}{a_i} \quad & = \quad 0, \text{ for } i = 1, 2. \end{aligned} \tag{6}$$

Optimal conditions for the co-state variables yield the following differential equations:

$$\dot{\lambda}_i(t) = [\delta + c_i] \, \lambda_i(t) - c_j \lambda_j(t) + \frac{\partial D}{\partial x_i} \text{ for } i, j = 1, 2; i = j. \tag{7}$$

For interior solutions, transversality conditions $\lambda_i(T) = -\partial D^T(x_i(T))/\partial x_i$, for $i = 1, 2$ imply

$$\frac{\mathrm{d}h_i(q_i(T))}{\mathrm{d}q_i} = \frac{\partial D(\mathbf{q}(T), \mathbf{x}(T))}{\partial q_i} + \frac{\partial D^T(\mathbf{x}(T))}{\partial x_i}, \quad \text{for } i = 1, 2, \tag{8}$$

providing terminal conditions for extraction rates and water table levels.

Conditions (6), (7), and (8) are necessary and sufficient for optimality (see Sethi and

Thompson [25]). Differentiating Eq. (6) with respect to t yields, for an interior solution,

$$\frac{\dot{\lambda}_i(t)}{a_i} = \left(\frac{\mathrm{d}^2 h_i}{\mathrm{d}q_i^2} - \frac{\partial^2 D}{\partial q_i^2} \right) \dot{q}_i(t) - \frac{\partial^2 D}{\partial q_i \partial q_j} \dot{q}_j(t) - \frac{\partial^2 D}{\partial q_i \partial x_i} \dot{x}_i(t) - \frac{\partial^2 D}{\partial q_i \partial x_j} \dot{x}_j(t). \tag{9}$$

Substituting Eqs. (6) and (9) into (7), and rewriting the stock dynamics given by Eq. (1), we obtain four differential equations involving $\mathbf{q}(t)$ and $\mathbf{x}(t)$. This system, together with initial conditions on the water stocks and terminal conditions on the extraction rates, specifies the socially optimal extraction and water stock paths $\langle \mathbf{q}^{SO}(t), \mathbf{x}^{SO}(t) \rangle$.

4 Policy Analysis

We suppose the regulator does not have resources to monitor agents' extraction decisions, but can monitor the state variables $\mathbf{x}(t)$. This scenario is analogous to a nonpoint source pollution problem in which the regulator can monitor ambient pollution levels but not individual emissions (see, for example, Segerson [24] and Xepapadeas [30]). The regulator is further restricted in that the only policy tools at her disposal are linear transfers, $\beta(t)$ for $t < T$ and β^T for $t = T$. In Theorem 1 below, we show that in spite of these restrictions, the regulator can induce the socially optimal path in Markov perfect equilibrium with the mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t) \equiv (\phi_1(x_1(t), t), \phi_2(x_2(t), t))'$, where

$$\phi_i(x_i(t), t) = \begin{cases} \beta_i(t)[x_i(t) - x_i^{SO}(t)] & \text{for } t < T \\ \beta_i^T [x_i(t) - x_i^{SO}(t)] & \text{for } t = T. \end{cases} \tag{10}$$

Before proving this result, it is useful to discuss Markov equilibria for linear mechanisms.

A mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t)$ induces a differential game between the agents. Given a strategy

$q_j^*(\mathbf{x}(t), t)$ chosen by agent j, agent i chooses as his strategy the solution to:

$$
\begin{aligned}
\max_{q_i(t)} \quad & \int_0^T \mathrm{e}^{-\delta t}[h_i(q_i(t)) + \beta_i(t)[x_i - x_i^{SO}]\mathrm{d}t + \mathrm{e}^{-\delta T}\beta_i^T[x_i(T) - x_i^{SO}(T)] \\
\text{subject to:} \quad & a_i\dot{x}_i(t) = r - q_i(t) + c[x_j(t) - x_i(t)] \\
& a_j\dot{x}_j(t) = r - q_j^*(\mathbf{x}(t), t) + c[x_i(t) - x_j(t)] \\
& q_i(t) \geq \mathbf{0}, \quad \mathbf{x}(0) = \mathbf{x}^0.
\end{aligned} \tag{11}
$$

An open-loop strategy is one in which agents pre-commit to an entire extraction path at the beginning of the game, and so is not a function of current state variables. Formally, a strategy $q_i(\mathbf{x}(t), t)$ is open-loop, if $q_i^*(\mathbf{x}(t), t) = q_i^*(t)$ for all $\mathbf{x}(t) \in \Re_+^2$. An open-loop Nash equilibrium (defined below) is relatively simple to compute for this game.

Definition 1 *A set $(q_1^*(t), q_2^*(t))$ of open-loop strategies where $q_i^*(t) : [0, T] \mapsto \Re$, is an open-loop Nash equilibrium if, for each $i \in \{1, 2\}$ an optimal control path $q_i(t)$ of the maximization problem given by (11) exists and is given by $q_i(t) = q_i^*(t)$.*

In general, open-loop Nash equilibria are restrictive since they do not allow agents to adapt strategies to changes in the state vector in a way that maximizes their current payoffs. This equilibrium concept is typically justifiable only in instances where the state vector is not observable over time, rendering moot the ability to adapt.

Markov-perfect equilibrium (defined below) is an alternative concept that overcomes this shortcoming. An agent choosing a Markov strategy conditions his current extraction only on the value of the current state variable (not otherwise on the game's previous history).

Definition 2 *Let $\mathbf{x}(t) \in X \subseteq \Re_+^2$ for all $t \in [0, T]$. A set $(q_1^*(\mathbf{x}(t), t), q_2^*(\mathbf{x}(t), t))$ of Markovian strategies where $q_i^*(\mathbf{x}(t), t) : X \times [0, T] \mapsto \Re$, is a Markovian-Nash equilibrium if, for each $i \in \{1, 2\}$ an optimal control path $q_i(t)$ of the maximization problem given by (11) exists and is given by $q_i(t) = q_i^*(\mathbf{x}(t), t)$. A Markov-perfect equilibrium is a subgame-perfect Markovian-Nash equilibrium.*

Identifying a Markov-perfect equilibrium typically requires the solution of a complex system of Hamilton-Jacobi-Bellman equations. The game considered here, however, has a structure that simplifies calculation of Markov-perfect equilibria. Specifically, it is a linear state game (as defined by Dockner et al. [9]) since (a) its objective functionals and state dynamics are linear in the state and (b) there are no cross terms of the sort $q_i x_i$ involving control and state variables. Dockner et al. [9] (pp. 187-89) show that all open-loop Nash equilibria of linear state games are Markov-perfect.

The following proposition characterizes a Markov-perfect equilibrium induced by the linear mechanism described above.

Proposition 1 *The differential game (11) induced by mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t)$ has a unique Markov-perfect equilibrium in open loop strategies $\mathbf{q}^*(t)$ that satisfies:*

$$\frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} - \frac{f_i^{\boldsymbol{\phi}}(t)}{a_i} \leq 0 \tag{12}$$

$$q_i^*(t) \quad \frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} - \frac{f_i^{\boldsymbol{\phi}}(t)}{a_i} \quad = 0; \text{ where} \tag{13}$$

$$f_i^{\boldsymbol{\phi}}(t) = \int_t^T \beta_i(s) \mathrm{e}^{\delta[t-s]} \frac{a_i + a_j \mathrm{e}^{[c_1+c_2][t-s]}}{a_1 + a_2} \mathrm{d}s + \beta_i^T \mathrm{e}^{\delta[t-T]} \frac{a_i + a_j \mathrm{e}^{[c_1+c_2][t-T]}}{a_1 + a_2}. \tag{14}$$

Proof. Let $\boldsymbol{\lambda}_i(t) = (\lambda_i^1, \lambda_i^2)'$ denote the costate variables for agent i corresponding to the state variables for agents 1 and 2. For an open-loop Nash equilibrium, the current-value Hamiltonian of agent i is:

$$\begin{aligned} H_i(q_i(t), \mathbf{x}(t), \boldsymbol{\lambda}_i(t), t) \quad = \quad & h_i(q_i) + \beta_i(t)[x_i(t) - x_i^{SO}(t)] + \\ & \sum_{k=1, j=k}^{2} \frac{\lambda_i^k(t)\,[r - q_k(t) + c[x_j(t) - x_k(t)]]}{a_k}. \end{aligned} \tag{15}$$

The necessary conditions for maximization for $i = 1, 2$ and $j = i$ are:

$$
\begin{aligned}
\frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} - \frac{\lambda_i^i(t)}{a_i} &\leq 0, \\
q_i^*(t) \quad \frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} - \frac{\lambda_i^i(t)}{a_i} &= 0, \text{ and} && (16) \\
\dot{\lambda}_i^i(t) &= [\delta + c_i]\lambda_i^i(t) - c_j\lambda_j^i(t) - \beta_i(t) && (17) \\
\dot{\lambda}_i^j(t) &= [\delta + c_j]\lambda_i^j(t) - c_i\lambda_i^i(t), && (18)
\end{aligned}
$$

with transversality conditions

$$
\lambda_i^i(T) = \beta_i^T, \quad \lambda_i^j(T) = 0. \tag{19}
$$

Since $H_i(\cdot)$ is jointly concave in $q_i(t)$ and $\mathbf{x}(t)$, these conditions are sufficient. Eqs. (17) and (18) are a linear system of ordinary differential equations. Imposing the transversality condition yields the unique solution for $i = 1, 2$ and $j = i$:

$$
\begin{aligned}
\lambda_i^i(t) &= f_i^{\boldsymbol{\phi}}(t), && (20) \\
\lambda_i^j(t) &= \int_t^T \frac{\beta_i(s)\mathrm{e}^{\delta[t-s]} \quad 1 - \mathrm{e}^{[c_1+c_2][t-s]} \quad a_j}{a_1 + a_2}\mathrm{d}s + \frac{\beta_i^T \mathrm{e}^{\delta[t-T]} \quad 1 - \mathrm{e}^{[c_1+c_2][t-T]} \quad a_j}{a_1 + a_2}. && (21)
\end{aligned}
$$

Substituting $\lambda_i^i(t)$ into Eq. (16) obtains the desired result. Uniqueness follows from the assumption that h_i is strictly concave. ■

To interpret Proposition 1, it is useful to calculate the shadow value of a unit of water table depth for agent i at time t given a mechanism $\boldsymbol{\phi}(\mathbf{x}(t), t)$. Eq. (2) implies for $s \in (t, T)$,

$$
\begin{aligned}
x_i(s) = \frac{1}{a_1 + a_2} \quad & x_i(t) \quad a_i + a_j\mathrm{e}^{[c_1+c_2][t-s]} \quad + x_j(t)a_j \quad 1 - \mathrm{e}^{[c_1+c_2][t-s]} \quad + && (22) \\
\int_t^s \quad & [r - q_i(z)] \quad 1 + \frac{a_j}{a_i}\mathrm{e}^{[c_1+c_2][z-s]} \quad + [r - q_j(z)] \quad 1 - \mathrm{e}^{[c_1+c_2][z-s]} \quad \mathrm{d}z \quad , && (23)
\end{aligned}
$$

so

$$
\frac{\partial}{\partial x_i(t)} \int_t^T x_i(s)\mathrm{d}s = \int_t^T \frac{a_i + a_j\mathrm{e}^{[c_1+c_2][t-s]}}{a_1 + a_2} \mathrm{d}s. \tag{24}
$$

For $\boldsymbol{\phi}(\mathbf{x}(t), t)$, the price for each unit of $x_i(s)$ at time s is $\beta_i(s)$, and the price of x_i^T at time T is β_i^T. The shadow price, or present discounted discounted value (at time t) of the stream of losses incurred from a marginal drop in the water table at time t is then $f_i^{\boldsymbol{\phi}}(t)$ in Eq. (14). To convert the shadow value from a marginal change in depth, x, to a marginal change in volume, q, it is necessary to divide by a. Thus, Eq. (12) states that in equilibrium agents set marginal profit from extraction equal to its shadow value.

For an isolated aquifer ($c = 0$), the term $[a_i + a_j]\mathrm{e}^{[c_1+c_2][t-s]}/[a_1 + a_2]$ reduces to unity, i.e., the full impact of extraction is on x_i. For the bathtub case ($c \to \infty$), it reduces to $a_i/[a_1 + a_2]$: The impact is proportional to the agent's relative share of the aquifer.

We now show that linear-state mechanisms can induce every feasible and continuously differentiable extraction path over $[0, T]$.

Theorem 1 *Let $\left\{\hat{q}_i(t) : t \in [0, T]\right\}$, be an arbitrary continuously differentiable feasible extraction path satisfying $\frac{\mathrm{d}h_i(\hat{q}_i(t))}{\mathrm{d}q_i} < \infty$, and $q_i^{\boldsymbol{\phi}}(t)$ be the unique open-loop Markov-perfect equilibrium extraction path induced by linear-state mechanism $\boldsymbol{\phi}$. (i) There exists a mechanism such that $q_i^{\boldsymbol{\phi}}(t) = \hat{q}_i(t)$ for all $t \in [0, T]$ and $i = 1, 2$. (ii) If $q_i^{\boldsymbol{\phi}}(t)$ is everywhere interior, then ϕ_i is unique.*[7]

Proof. Suppose $\hat{q}_i(t)$ is everywhere interior. Since $h_i(\cdot)$ is strictly concave, it is sufficient to show that for any $\hat{q}_i(t)$, terms $\beta_i(t)$ and β_i^T of mechanism ϕ_i can be chosen such that

$$\frac{\mathrm{d}h_i(\hat{q}_i(t))}{\mathrm{d}q_i} = \frac{f_i^{\boldsymbol{\phi}}}{a_i}, \quad \text{for all } t \in [0, T]. \tag{25}$$

Suppose $\beta_i^T = a_i \mathrm{d}h_i(\hat{q}_i(T))/\mathrm{d}q_i$, ensuring that Eq. (25) is satisfied for $t = T$. Eq. (25) becomes

$$\int_t^T \beta_i(s)\mathrm{e}^{\delta[t-s]}\frac{a_i + a_j\mathrm{e}^{[c_1+c_2][t-s]}}{a_1 + a_2}\mathrm{d}s \;=\; a_i\frac{\mathrm{d}h_i(\hat{q}_i(t))}{\mathrm{d}q_i} - \beta_i^T\mathrm{e}^{\delta[t-T]}\frac{a_i + a_j\mathrm{e}^{[c_1+c_2][t-T]}}{a_1 + a_2}. \tag{26}$$

[7] A somewhat different technical argument ensures that the essence of Theorem 1 extends to the case where agents incur energy costs in the sense of Gisser and Sanchez [12]. In such instances a more complex, nonlinear mechanism is required to induce the target path in MPE. The details are presented in the Appendix.

Performing the change of variable $z = T - t$, we have

$$-\int_0^z \beta_i(T-s)\mathrm{e}^{\delta[s-z]}\frac{a_i + a_j\mathrm{e}^{[c_1+c_2][s-z]}}{a_1+a_2}\mathrm{d}s \;=\; a_i\frac{\mathrm{d}h_i(\hat{q}_i(T-z))}{\mathrm{d}q_i} - \beta_i^T\mathrm{e}^{-\delta z}\frac{a_i + a_j\mathrm{e}^{-[c_1+c_2]z}}{a_1+a_2}. \tag{27}$$

Eq. (27) is a linear Volterra equation of the first kind with a kernel containing exponential functions, a general solution for which can be found in Polyanin and Manzhirov [19](p. 17). In our case, letting

$$g(z) = -a_i\frac{\mathrm{d}h_i(\hat{q}_i(T-z))}{\mathrm{d}q_i} + \beta_i^T\mathrm{e}^{-\delta z}\frac{a_i + a_j\mathrm{e}^{-[c_1+c_2]z}}{a_1+a_2},$$

the solution to (27) is given by

$$\beta_i(T-z) \;=\; \mathrm{e}^{-\delta z}\frac{\mathrm{d}}{\mathrm{d}z}\; \mathrm{e}^{-\frac{a_i[c_1+c_2]z}{a_1+a_2}} \int_0^z \frac{\mathrm{d}}{\mathrm{d}s}\; g(s)\mathrm{e}^{[\delta+c_1+c_2]s}\; \mathrm{e}^{\frac{a_j[c_1+c_2]s}{a_1+a_2}}\mathrm{d}s \;. \tag{28}$$

The assumed differentiability of $h_i(\cdot)$ and $\hat{q}_i(\cdot)$ ensures that Eq. (28) is well-defined. Repeating this argument for agent $j = i$ and collecting the $\beta_i(\cdot), \beta_j(\cdot)$ functions and β_i^T, β_j^T constants establishes the desired result. Uniqueness under an interior path follows from the fact that conditions (25) for $i \in \{1, 2\}$ are, in this case, sufficient for the two paths to coincide.

If $\hat{q}(t)$ is not interior everywhere, imposing Eq. (25) at boundary points forces the equilibrium conditions (16) to set the appropriate boundary value. ■

We conclude this section with the following corollary to Theorem 1.

Corollary 1 *If $\mathbf{q}^{SO}(t)$ is continuously differentiable there exists a linear mechanism that induces it in Markov-perfect equilibrium with zero net transfers.*

The corollary follows directly from Theorem 1. If the socially optimal extraction path is continuously differentiable, and the regulator induces it with a mechanism $\boldsymbol{\phi}^{SO}$, then $\mathbf{x}(t) = \mathbf{x}^{SO}(t)$ for all t, and, by Eq. (10), $\boldsymbol{\phi}^{SO}(\mathbf{x}^{SO}(t), t) = \mathbf{0}$ for all t.

5 Numerical Simulations

In this section, we numerically simulate the differential game (11) for two agricultural agents in a typical rural setting in semi–arid tropical India. In these regions, agricultural production was traditionally constrained by precipitation variations during the wet monsoon season. The advent of inexpensive pump technology in the 1970s coupled with subsidized electricity now allows year-round production [26, 21].

Table 1 lists the parameters used in the simulation (see Raj [20] for data on climate and groundwater, Kijne et al. [16] for crop and agricultural production specific data, and a 2001 World Bank report [29] plus references therein for energy data). We calculate monetary units in 2005 U.S. dollars, using the average annual exchange rate. Farmers are adjacent landholders with one hectare plots. They share a watershed that receives no recharge through lateral subsurface inflows over the boundary. As described in Section 2, a hydraulic connection between the adjacent landholdings allows water to flow across this interface depending on the individual water table elevations. We assume homogeneous and isotropic aquifer properties and choose parameter values representative of subsurface properties of weathered crystalline rock found in large parts of peninsular India. We suppose constant characteristic values for the hydraulic transmissivity c. For both farmers, the water table at $t = 0$ is at $\hat{x} = 0$ meters above sea level.

The agro-economic parameters are representative of small landholders growing paddy rice in two seasons per year. Each farmer pumps water from one borehole located on his plot. We specify a quadratic restricted profit function for agents $i = 1, 2$:

$$h_i(q_i) = \theta_i \left(\alpha_1 q_i + \alpha_2 q_i^2 \right), \tag{29}$$

that implicitly assumes rainfed agricultural production is infeasible [13]. Panel 1 of Figure 1 illustrates profit functions for both farmers. Farmer 1 is more technically efficient in the sense that he can attain any feasible profit using less water than Farmer 2.

Parameter	Description	Value	Unit
A	landholding size	1	ha
S	effective porosity	0.01	n.a.
c	transmissivity	3.3×10^{-2}	m^2/day
$\bar{x}$	top of aquifer	0	m
r	mean daily recharge	2.2×10^{-4}	m/day
δ	discount rate	0.03	n.a.
θ_1	technological parameter	1	US\$ day / m^3
θ_2	technological parameter	0.9	US\$ day / m^3
α_1	technological parameter	2.5	US\$ day / m^6
α_2	technological parameter	-0.0312	US\$ day / m^6
d_1	fixed energy cost parameter	0.1	US\$ day / m^3
d_2	variable energy cost parameter	0.01	US\$ day / m^4

Table 1: Simulation parameter values

Social costs reflect typical expenses for the state related to provision of rural energy and are presumed to be the same for both farmers. The energy cost function [12, 22] is:

$$D(\mathbf{q}, \mathbf{x}) = \sum_{i=1}^{2} q_i[d_1 + d_2[\bar{x} - x_i]], \tag{30}$$

where $\bar{x}$ denotes the elevation of the irrigated plot. Here, $\bar{x} - x_i$ is the the total drawdown for each agent at any given moment in time against which a certain quantity of water has to be lifted to the surface. We set the terminal time cost to $D^T(\mathbf{x}(T)) = 0$.

For all computations, we use Matlab with Simulink. We solve the system of differential equations (5) as a nonlinear programming problem using the control vector parameterization concept described in [27] and the references therein. We utilize the Dormand-Prince formula fixed-step integration technique solver to obtain the socially optimal solution, with a daily discretization time-step and $T = 365$.

Simulation results are shown in Figures 1 and 2. Socially optimal (SO) pumping rates decline over time (Panel 2 in Figure 1). Privately optimizing (PO) pumping rates, representing the outcome of the unregulated status quo, are constant throughout the optimization period since extraction costs are not internalized by the agents.[8] Panel 3 illustrates how

[8] In this context, agents set their pumping rates to q_i such that $\mathrm{d}h_i(q_i)/\mathrm{d}q_i = 0$.

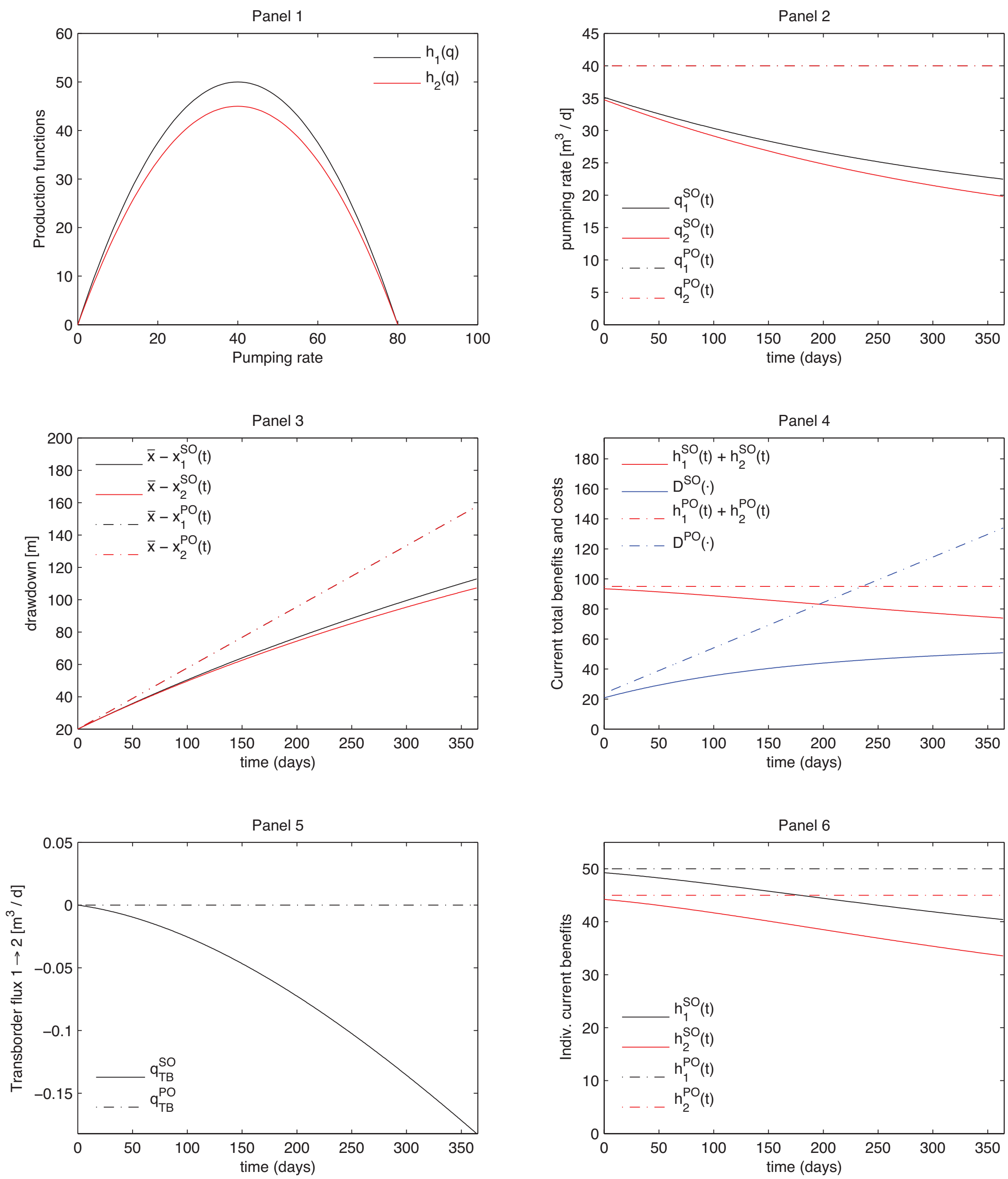

Figure 1: Simulation results. SO and PO denote socially and privately optimal extraction.

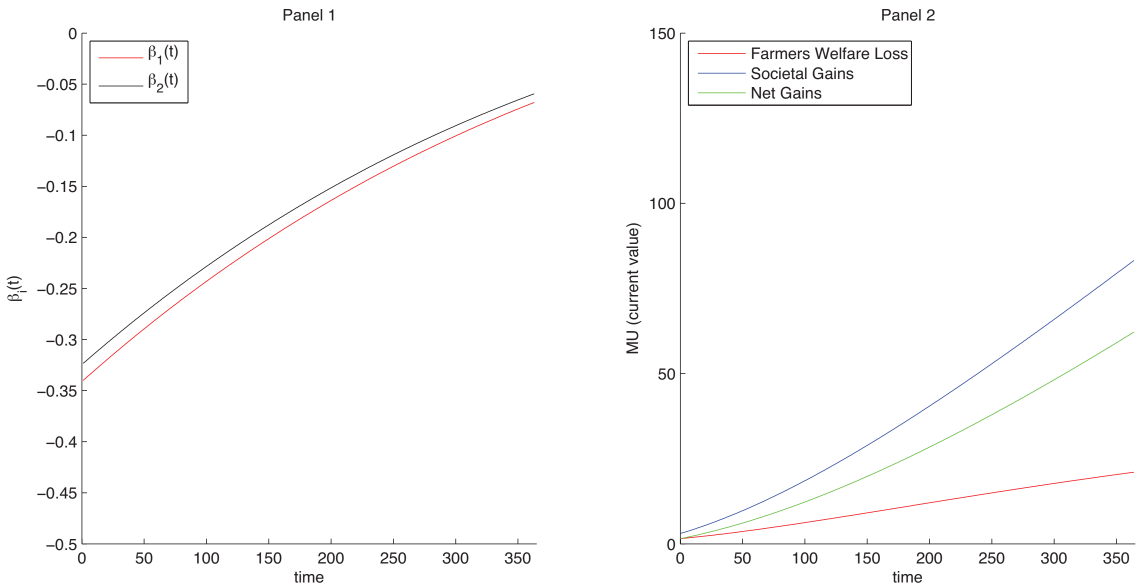

Figure 2: Simulated welfare effects

water tables decline at a slower rate at the social optimum, thus resulting in lower social damages (Panel 4). Panel 5 shows the transborder flux in the micro-watershed between the two adjacent landholdings for both runs. A positive flux indicates a net subsurface water exchange from west to east and vice versa. The particular socially optimal pumping dynamics induce a reversal of the transborder flux as soon as $q_1^{SO}(t) > q_2^{SO}(t)$ (compare Panels 2 and 3). Since farmer 2 always pumps at higher rates as compared to farmer 1, no such flux reversal is visible for the competitive equilibrium solutions where the micro-watershed of the more efficient farmer 1 (see also Panel 1) loses water to the less efficient eastern neighbor throughout the simulated period.

Panel 1 in Figure 2 shows the development of the mechanism charges $\beta_1(t)$ and $\beta_2(t)$ to farmers 1 and 2, respectively, as a function of time with $t < T$. For the terminal time charges, we have $\beta_1^T = 109.5$ and $\beta_2^T = 113.5$. Panel 1 suggests a surprising result: Dynamic mechanism charges are (in this case) negative, implying that agents are, in theory, rewarded when water table levels are below socially optimal levels. This counter-intuitive result, which has been observed in other contexts (see Benchekroun and Long [4]), can be explained in two ways. First, these charges are minute compared to their terminal-time counterpart. The latter is about three orders of magnitude greater, and positive, and dominates agent behavior despite future discounting. Second, as argued in Benchenkroun and Long [4], even though agents would be receiving a subsidy for excessive pumping, they refrain from over-extracting because they anticipate that if they do so, the subsidy will become smaller.

This discussion leads to a natural research question. As monitoring water tables across time is costly and can result in strange policy recommendations, it is worth carefully exploring the effects of only imposing a terminal-time charge on agents' water tables. If welfare loss is not excessive, such a restriction on policy may be desirable.

The red line in Panel 2 of Figure 2 shows total profit losses to farmers (in comparison to the unregulated status quo) due to their pumping at the socially optimal levels. These losses correspond to the difference between the red dotted and solid lines in Panel 4 of Figure 1. The blue line in Panel 2 of Figure 2 shows the social cost savings achieved from reverting to socially optimal extraction. These savings correspond to the difference between the blue dotted and solid lines in Panel 4 in Figure 1. Throughout the simulation period, social cost savings are higher than farmer profit losses, with their net difference depicted by the green line in Panel 2 of Figure 2. Hence, part of the overall social benefits can be redistributed to farmers to compensate their welfare losses. Such compensation may be important from the perspective of a real-world implementation since without it, farmers would not acquiesce to public policy of this sort unless they were coerced to do so.

5.1 The Role of Spatial and Economic Complexity

In this section we discuss welfare loss from making two kinds of mistakes in implementing the policy outlined in Section 4: (a) incorrectly assuming that the underlying aquifer is a bathtub, or; (b) incorrectly assuming that agents are homogeneous.

Under assumption (a), the regulator solves for the socially optimal extraction path assuming incorrectly that the aquifer has infinite transmissivity. In particular, she solves the optimal control problem given by Expression (5) with state dynamics given by Lemma 1.

Then, she plugs the derived extraction path into Eq. (28), assuming that $c_1 = c_2 = \infty$, to obtain the mechanism charges. Implementation of this mechanism results in an induced equilibrium described by Proposition 1.

Figure 3 shows that aquifer dynamics have a major effect on optimal policy when transmissivity is low, a feature commonly found in hard rock or well consolidated sedimentary formations. The graph depicts percentage welfare loss (with regard to the social optimum) as a function of actual field transmissivity values c. The range over which c is varied ($2\times10^{-4}-10$ m^2/s) corresponds to field situations as reported in Raj [20]. Obviously, the negative impact increases the less the aquifer resembles a bathtub in reality. As transmissivity increases, the bathtub assumption results in less welfare loss and eventually becomes innocuous.

Turning to assumption (b), economic heterogeneity is defined as the ratio between the value of the two profit functions in Eq. (29). In our calculations, we take h_1 as given and vary θ_2 from 0.1 to 1. The regulator's mistake is now the following. First, she solves for the socially optimal extraction path assuming incorrectly that the two agents have identical profit functions. That is, she solves the optimal control problem given by Expression (5) supposing that $\theta_2 = 1$ in the objective function. Then, she plugs the derived extraction path into Eq. (28) to obtain the mechanism charges. The subsequent implementation of this mechanism results is an induced equilibrium, described by Proposition 1, that is suboptimal in relation to the social optimum, which is the solution to optimal control problem (5) with the correct value of θ_2.

Figure 4 depicts percentage welfare loss (with regard to the social optimum) as a function of actual heterogeneity. The simulation suggests that adverse welfare impacts increase with agent heterogeneity, potentially reaching high levels.

6 Conclusion

This paper develops a differential game-theoretic model of groundwater extraction accounting for spatial and economic complexity. Agents have different payoff functions and affect

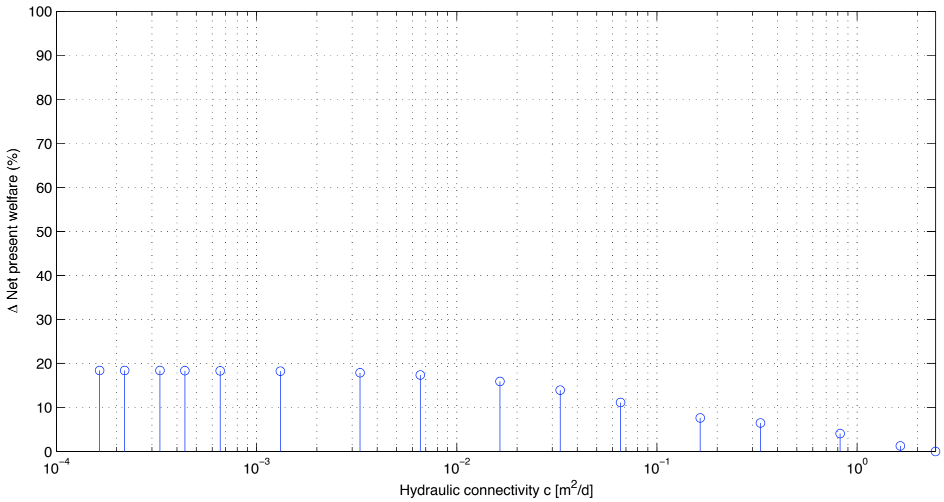

Figure 3: Percent social welfare loss from incorrect "bathtub" assumption.

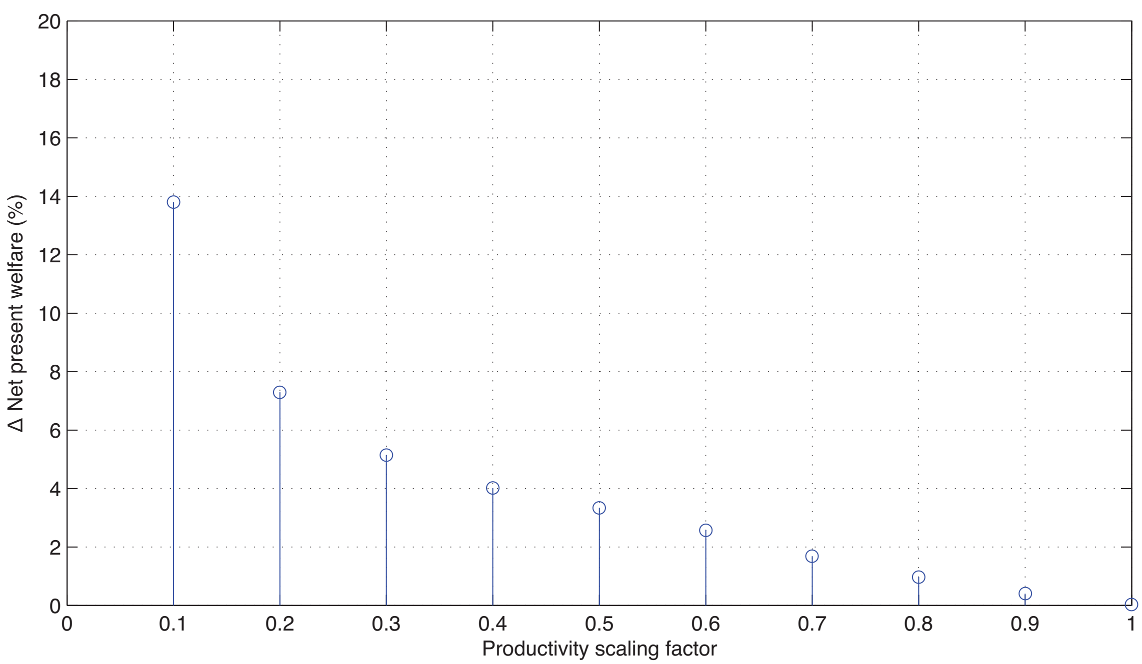

Figure 4: Percent social welfare loss from incorrect farm homogeneity assumption.

each other to an arbitrary degree, thus generalizing the canonical one-dimensional common-property paradigm. Focusing on the two-cell finite-horizon case, we study simple linear incentive mechanisms and analyze the induced Markov-perfect equilibrium behavior. We find a linear-state mechanism that induces the socially optimal extraction path. The theoretical analysis extends to general multi-cell environments. Simulations suggest that our approach holds promise in ameliorating inefficient energy and groundwater use patterns in rural India.

Appendix

Extension to Multiple Agents. Let there be n agents indexed by $i = 1, 2, ..., n$, and let $S_i \subseteq \{1, 2, .., n\} \equiv S$ denote agent i's neighbors. The degree to which neighboring agents i and j are connected is denoted by $c_{ij} = c_{ji}$ where $c_{ij} \in [0, \infty]$ (we set $c_{ii} = 0$). Generalizing the dynamics of Eq. (1), the water table of agent of agent i obeys the following differential equation[9]

$$a_i \dot{x}_i = r - q_i + \sum_{j \in S_i} c_{ij}[x_j - x_i]. \tag{31}$$

In a multiple-agent environment, the vectors $\mathbf{x}$ and $\mathbf{q}$ are extended in the obvious way and the social cost function is generalized to $D(\mathbf{q}, \mathbf{x})$. The socially optimal solution given by the optimization problem (5) is also appropriately modified.

In the context of the linear mechanisms that were discussed in Section 4 it is possible to adapt Proposition 1 to give us insight into the equilibrium behavior of the system. In particular, given an agent i and a mechanism $\boldsymbol{\phi}$, his Markov-perfect open loop equilibrium extraction path q_i^* satisfies

$$\begin{aligned} \frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} - \frac{\lambda_i^{i,\boldsymbol{\phi}}(t)}{a_i} &\leq 0 \\ q_i^*(t) \quad \frac{\mathrm{d}h_i(q_i^*(t))}{\mathrm{d}q_i} - \frac{\lambda_i^{i,\boldsymbol{\phi}}(t)}{a_i} &= 0. \end{aligned}$$

[9]When $c_{ij} = \infty$ agents i and j share the same cell and their interaction is described by the multiple-agent equivalent of the dynamics that appear in Footnote 5. To avoid cumbersome notation we remain consistent with our previous analysis and assume that all n agents are in different cells.

where $\lambda^{i,\phi}(t)$ solves the following system of differential equations with terminal conditions:

$$\dot{\boldsymbol{\lambda}}^{\boldsymbol{i,\phi}} = \mathbf{A} \cdot \boldsymbol{\lambda}^{i,\phi}(t) + \mathbf{b}^i(t)$$
$$\lambda_i^{i,\phi}(T) = \beta_i^T, \;\; \lambda_j^{i,\phi}(T) = 0 \;\text{ for all } j = i. \tag{32}$$

Here $\mathbf{A} \in \Re^{n\times n}$ and $\mathbf{b}^i \in \Re^n$ are such that

$$\mathbf{A}_{kk} = \delta + \frac{1}{a_k}\sum_{j\in S_k} c_{kj} \;\text{ for all } k \in \{1,2,..,n\}$$
$$\mathbf{A}_{kj} = -\frac{c_{kj}}{a_j}, \quad \text{for all } \; k \in \{1,2,...,n\}, \;\; j \in S_k$$
$$\mathbf{A}_{kj} = 0, \quad \text{for all } \; k \in \{1,2,...,n\}, \;\; j \notin S_k$$
$$\mathbf{b}_i^i = -\beta_i(t), \;\; \text{and } \; \mathbf{b}_j^i = 0 \;\text{ otherwise.}$$

Once a solution to system (32) is obtained, it can be used to influence Markov-perfect equilibrium behavior along the lines of Theorem 1. In particular, given an arbitrary feasible, continuously differentiable, n-dimensional extraction path, we may apply the results in Athanassoglou [2] to establish that a linear-state mechanism exists, which induces it in Markov-perfect equilibrium.

Theorem 2 *Theorem 1 extends to the n-agent case.*

Proof. Similarly to the proof of Theorem 1, we wish to find $\boldsymbol{\phi}$ so that

$$\frac{\mathrm{d}h_i(\hat{q}_i(t))}{\mathrm{d}q_i} = \frac{\lambda_i^{i,\phi}(t)}{a_i}, \;\text{ for all } t \in [0,T]. \tag{33}$$

A general solution for the system of differential equations given by Eqs. (32) can be found in Chapter 2.3.4 of Coddington and Carslon [8] and is the following

$$\boldsymbol{\lambda}^{i,\phi}(t) = \boldsymbol{\Lambda}^i(t)\boldsymbol{\xi} + \boldsymbol{\Lambda}^i(t) \int_0^t \boldsymbol{\Lambda}^i(s)^{-1}\mathbf{b}^i(s)\mathrm{d}s, \;\; t \in [0,T] \tag{34}$$

where $\boldsymbol{\xi} \in \Re^n$ and $\boldsymbol{\Lambda}^i(t)$ is a basis for the solutions to the homogeneous counterpart of system (32). Performing the change of variable $z = T - t$, choosing $\boldsymbol{\Lambda}^i$ so that $\boldsymbol{\Lambda}^i(z) = \mathbf{I}_n$ at

$z = 0$, and setting $\boldsymbol{\xi}$ to a vector $\boldsymbol{\xi}^{\beta_i^T}$ such that the transversality conditions in Eqs. (32) are satisfied,[10] obtains the following unique solution of system (32)

$$\boldsymbol{\lambda}^{i,\boldsymbol{\phi}}(z) = \boldsymbol{\Lambda}^i(z)\boldsymbol{\xi}^{\beta_i^T} - \boldsymbol{\Lambda}^i(z) \int_0^z \left[\boldsymbol{\Lambda}^i(s)\right]^{-1}\mathbf{b}^i(T-s)\mathrm{d}s, \quad z \in [0,T]. \tag{35}$$

Denote row j of matrix $\boldsymbol{\Lambda}^i$ by $\boldsymbol{\Lambda}^i_j$. The restriction of vector (35) to coordinate i obtains

$$\boldsymbol{\lambda}_i^{i,\boldsymbol{\phi}}(z) = \boldsymbol{\Lambda}_i^i(z)\boldsymbol{\xi}^{\beta_i^T} - \int_0^z \left(\boldsymbol{\Lambda}^i(z)\left[\boldsymbol{\Lambda}^i(s)\right]^{-1}\right)_{ii}\beta_i(T-s)\mathrm{d}s, \quad t \in [0,T], \quad z \in [0,T]. \tag{36}$$

Using Eq. (36), we adapt condition (26) to obtain the following Volterra integral equation of the first kind

$$\begin{aligned} -a_i \frac{\mathrm{d}h_i(\hat{q}_i(T-z))}{\mathrm{d}q_i} + \boldsymbol{\Lambda}_i^i(z)\boldsymbol{\xi}^{\beta_i^T} = \\ \int_0^z \left(\boldsymbol{\Lambda}^i(z)\left[\boldsymbol{\Lambda}^i(s)\right]^{-1}\right)_{ii}\beta_i(T-s)\mathrm{d}s, \quad \text{for all } z \in [0,T]. \end{aligned} \tag{37}$$

We set β_i^T so that Eq. (38) is satisfied for $z = 0$. The integral equation's kernel

$$\Theta(z,s) = \left(\boldsymbol{\Lambda}^i(z)\left[\boldsymbol{\Lambda}^i(s)\right]^{-1}\right)_{ii}$$

is such that $\Theta(z,z) = 1$. This fact, in combination with our differentiability assumptions, implies that Eq. (38) may be reduced to the following equivalent Volterra integral equation of the second kind

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left(-a_i \frac{\mathrm{d}h_i(\hat{q}_i(T-z))}{\mathrm{d}q_i} + \boldsymbol{\Lambda}_i^i(z)\boldsymbol{\xi}^{\beta_i^T}\right) = \\ \beta_i(T-z) + \int_0^z \frac{\mathrm{d}}{\mathrm{d}z}\Theta(z,s)\beta_i(T-s)\mathrm{d}s, \quad z \in [0,T]. \end{aligned} \tag{38}$$

Our continuity and differentiability assumptions ensure that Theorem 2.1.1 in Burton [6] applies and integral equation (39) has a unique solution. ∎

[10] Since the matrix $\boldsymbol{\Lambda}^i$ has full rank, $\boldsymbol{\xi}^{\beta_i^T}$ exists and is uniquely determined.

Extension to model with energy costs. Suppose agent i incurs an energy cost proportional to his water table depth, i.e., $\alpha_i q_i(x^0 - x_i)$ for some $\alpha_i > 0$. Here, x^0 refers to the water table level of a "full" aquifer, i.e., one which corresponds to a negligible drawdown. The profit function for agent i is

$$h_i(q_i) - \alpha_i q_i \;\; x^0 - x_i \;\; .$$

A special case of the above objective function (with h_i quadratic) is used in [12] and [22], among others. We show a nonlinear mechanism of the sort

$$\begin{aligned}
\phi_i(\mathbf{x}, t) &= \beta_i^1 \;\; x_i - \hat{x}_i(t) \;\; ^2 + \beta_i^2 \;\; x_i - \hat{x}_i(t) \;\; [x_j - \hat{x}_j(t) \\
&\quad + \beta_i^3(t) \;\; x_i - \hat{x}_i(t) \;\; + \beta_i^4(t) \;\; x_j - \hat{x}_j(t) \\
\phi_i^T(\mathbf{x}) &= \beta_i^{T,1} \;\; x_i - \hat{x}_i(T) \;\; ^2 + \beta_i^{T,2} \;\; x_i - \hat{x}_i(T)
\end{aligned}$$

for $i = 1, 2$ is able to induce, in Markov-perfect equilibrium, an arbitrary extraction path $\hat{\mathbf{q}}$ satisfying the conditions of Theorem 1.

The following argument is based on results in Athanassoglou [3]. In contrast to the analysis of Theorem 1, we consider the Hamilton-Jacobi-Bellman sufficient conditions for a Markov perfect equilibrium, which appear in Theorem 4.4 of Dockner et al. [9]. Consider a mechanism $\hat{\phi}$ and the Hamilton-Jacobi-Bellman equation for agent i, assuming that his opponent uses the open-loop strategy $\hat{q}_j(t)$:

$$\begin{aligned}
\delta V^i(\mathbf{x}, t) &- \frac{\partial}{\partial t} V^i(\mathbf{x}, t) = \max_{q_i \geq 0} \quad h_i(q_i) - \alpha_i q_i [x^0 - x_i] + \hat{\phi}_i(\mathbf{x}, t) \\
&\quad + V^i_{x_i}(\mathbf{x}, t) \frac{1}{a_i} \;\; r - q_i + c[x_j - x_i] \;\; + V^i_{x_j}(\mathbf{x}, t) \frac{1}{a_j} \;\; r - \hat{q}_j(t) + c[x_i - x_j] \quad . \\
V^i(\mathbf{x}, T) &= \phi_i^T(\mathbf{x})
\end{aligned} \tag{39}$$

We show that a mechanism ϕ can be chosen so that agent i's best-response is to adopt open-loop strategy $\hat{q}_i(t)$. First, to ensure that $\hat{q}_i(t)$ maximizes the RHS of Eq. (39) we impose that the value function satisfy

$$V^i_{x_i}(\mathbf{x},t) = a_i \ \frac{\mathrm{d}h_i}{\mathrm{d}q_i}(\hat{q}_i(t)) - \alpha_i \ x^0 - x_i \quad . \tag{40}$$

Moreover, we guess that $V^i_{x_j}(\mathbf{x},t) = 0$. With these conditions, we obtain a candidate value function given by

$$V^i(\mathbf{x},t) = \ \ a_i\frac{\mathrm{d}h_i}{\mathrm{d}q_i}(\hat{q}_i(t)) - a_i\alpha_i x^0 \ \ x_i + \frac{a_i\alpha_i}{2}x_i^2 + \hat{A}_i(t), \tag{41}$$

where $\hat{A}_i(t)$ is a function yet to be determined. Substituting the above into the HJB equations (39) we obtain

$$\begin{aligned}
\delta \ \ a_i\tfrac{\mathrm{d}h_i}{\mathrm{d}q_i}(\hat{q}_i(t)) - a_i\alpha_i x^0 \ \ x_i + \tfrac{\delta a_i\alpha_i}{2}x_i^2 + \delta\hat{A}_i(t) - a_i\tfrac{\partial}{\partial t} \ \ \tfrac{\mathrm{d}h_i}{\mathrm{d}q_i}(\hat{q}_i(t)) \ \ x_i - \tfrac{\mathrm{d}}{\mathrm{d}t}\hat{A}_i(t) \\
= h_i(\hat{q}_i(t)) - \alpha_i\hat{q}_i(t)[x^0 - x_i] + \hat{\phi}_i(\mathbf{x},t) \\
+ a_i \ \ \tfrac{\mathrm{d}h_i}{\mathrm{d}q_i}(\hat{q}_i(t)) - \alpha_i(x^0 - x_i) \ \ \tfrac{r-\hat{q}_i(t)+c[x_j-x_i]}{a_i},
\end{aligned} \tag{42}$$

$$a_i\tfrac{\mathrm{d}h_i}{\mathrm{d}q_i}(\hat{q}_i(T)) - a_i\alpha_i x^0 \ \ x_i + \tfrac{a_i\alpha_i}{2}x_i^2 + \hat{A}_i(T) = \phi_i^T(\mathbf{x}). \tag{43}$$

Considering the HJB conditions (43), let $g^i(\mathbf{x},t)$ and $g^{i,T}(\mathbf{x})$ denote the functions implicitly defined by the following equations

$$\begin{aligned}
g^i(\mathbf{x},t) + \delta\hat{A}_i(t) - \frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_i(t) &= \hat{\phi}_i(\mathbf{x},t), \quad t < T \\
g^{i,T}(\mathbf{x}) + \hat{A}_i(T) &= \hat{\phi}_i^T(\mathbf{x}).
\end{aligned} \tag{44}$$

Note that g^i and $g^{i,T}$ are polynomial in $\mathbf{x}$. Thus, $g^i(\mathbf{x},t)$ and $g^{i,T}(\mathbf{x})$ will equal their Taylor expansion, with respect to $\mathbf{x}$, about any point in $\Re^2$. With this fact in mind, consider the mechanism given by the non-constant part of a Taylor expansion about points $(\hat{\mathbf{x}}(t),t)$ and

$\hat{\mathbf{x}}(T)$ respectively. That is, let

$$\begin{aligned}
\hat{\phi}_i(\mathbf{x},t) &= [x_i - \hat{x}_i(t)]g^i_{x_i}(\hat{\mathbf{x}}(t),t) + [x_j - \hat{x}_j(t)]g^i_{x_j}(\hat{\mathbf{x}}(t),t) + \quad [x_i - \hat{x}_i(t)]^2 \underbrace{g^i_{x_ix_i}(\hat{\mathbf{x}}(t),t)}_{constant} \\
&\quad +2[x_i - \hat{x}_i(t)][x_j - \hat{x}_j(t)] \underbrace{g^i_{x_ix_j}(\hat{\mathbf{x}}(t),t)}_{constant} + [x_j - \hat{x}_j(t)]^2 \underbrace{g^i_{x_jx_j}(\hat{\mathbf{x}}(t),t)}_{0} \quad \frac{1}{2} \\
\hat{\phi}^T_i(\mathbf{x}) &= [x_i - \hat{x}_i(T)]g^{i,T}_{x_i}(\hat{\mathbf{x}}(T)) + \frac{1}{2}[x_i - \hat{x}_i(T)]^2 g^{i,T}_{x_ix_i}(\hat{\mathbf{x}}(T)). \qquad (45)
\end{aligned}$$

This implies that

$$\begin{aligned}
g^i(\mathbf{x},t) &= \hat{\phi}_i(\mathbf{x},t) + g^i(\hat{\mathbf{x}}(t),t) \\
g^{i,T}(\mathbf{x}) &= \hat{\phi}^T_i(\mathbf{x}) + g^{i,T}(\hat{\mathbf{x}}(T)).
\end{aligned}$$

Eq. (44) then yields the following ordinary differential equation with terminal condition

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_i(t) + \delta\hat{A}_i(t) + g^i(\hat{\mathbf{x}}(t),t) = 0, \\
&\hat{A}_i(T) = -g^{i,T}(\hat{\mathbf{x}}(T)). \qquad (46)
\end{aligned}$$

Differential equation (46) yields a unique solution $\hat{A}_i(t)$. Substituting this into the value function given by Eq. (41), repeating this argument for agent $j = i$, and invoking Theorem 4.4 in Dockner et al. [9] concludes the proof.

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